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# Random Driving of Nonlinear Oscillator

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## Disclaimer

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**Abstract**

The equation for the evolution of the density distribution and its two phase-space point correlation for nonlinear oscillators under the inflence of an external random driving force are derived. The normal Fokker-Planck description is shown to be applicable to the average distribution. Small-amplitude and short-wavelength spatial fluctuations of density ("microstructure") are found to be the special effect of the random-driving, distinguishing it from incoherent noise. These density fluctuations are analyzed using the correlation function formalism.

## 1 Introduction

The influence of random forces on nonlinear oscillators is a common problem in the random processes theory, with many applications in all fields of science. For an individual particle experiencing a randomly applied force, the resulting particle motion has the characteristics of the Brownian motion (see, e.g. /1/). Assuming that the probability function of finding a particle in the phase space is independent of the initial phase space coordinates (Markovian process), the evolution equation of the particle distribution function is reduced to the Fokker-Planck equation. A similar but not identical problem is the effect of rf noise in high energy particle accelerators, where the dynamics of the particle distribution is equivalent to that of a nonlinear oscillator driven by the random statistically independent force, which is the same for all particles. Thus, one confronts a somewhat unusual problem of a random coherent driving force. It can be shown, however, that the average (over the ensemble of

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driving forces) distribution function satisfies the Fokker-Planck evolution equation which appears when each particle is affected independently (incoherent noise). Many extensive studies have been published (see /2-4/), where the Fokker-Planck equation was analyzed and solved using the averaging techiques in the small noise/fast oscillation regime. Due to the nonlinearity of the Hamiltonian system, the diffusion coefficients can be expressed in terms of the noise power spectrum at multiples of the oscillation frequency. The theory has been verified by numerical simulations /4/ and has also been cofirmed by experimental observations /5/. Indeed, for a small amplitude driving force, the response of the nonlinear oscillator concentrates mainly around the harmonics of this force at the frequency of these oscillations. The important point is that the amplitudes at different harmonic frequencies of the same, sufficiently long section of random signal, are statistically independent. Thus, to the zeroth order approximation, the coherently random driving force and the Brownian motion with statistically independent random forces produce the same results.

Beyond the zeroth order approximation, how do the individual density distributions differ from the average one? Will the density fluctuations be smoothed out by the random noise? What is the effect of the nonlinearity on fluctuations? In the present paper, we address these questions by studying the fluctuations in the ensemble of density distributions, which can be described by the correlation function in both phase space and time. We study the spatial spectrum of the fluctuations (same-time correlation function) in the limit of small noise/ large nonlinearity.

The plan of the paper is as follows. A model of a nonlinear oscillator with coherent driving force is introduced. After defining the correlation function, we obtain the self-contained description of fluctuations by deriving the evolution equation for the correlator. A solution of this evolution equation in the limit of small noise/fast oscillations is discussed in section 4. The conclusion is given in section 5.

## 2 Model.

We consider the general form of the Hamiltonian of nonlinear oscillator with a random driving force,

$$H = \frac{p^2}{2} + g(q) + h(q)\xi(t) \tag{1}$$

where $g(q)$ is an arbitrary nonlinear potential and $\xi(t)$ is for simplicity, yet without loss of generality, chosen to be the white noise, i.e.

$$\langle \xi(t)\xi(t')\rangle = \delta(t-t'). \tag{2}$$

In the absence of particle interactions (or collisions), the evolution of the density distribution is governed by the Vlasov equation:

$$\frac{\partial f}{\partial t} - \left(\frac{\partial g}{\partial q} + \frac{\partial h}{\partial q}\xi(t)\right)\frac{\partial f}{\partial p} + p\frac{\partial f}{\partial q} = 0. \tag{3}$$

The statistical properties of the fluctuating quantity $f$ are appropriately defined by the ensemble average of the distribution function,

$$\bar{f}(p,q,t) = \langle f(p,q,t)\rangle_{\{\xi\}} \tag{4}$$

and the correlation function of the density fluctuations in adjacent phase space points,

$$K(p,q,\tilde{p},\tilde{q},t) = \langle (f(p,q,t) - \bar{f}(p,q,t))(f(\tilde{p},\tilde{q},t) - \bar{f}(\tilde{p},\tilde{q},t))\rangle_{\{\xi\}}\,. \tag{5}$$

We limit ourselves by considering only the same-time correlator $K$ and study therefore only the spatial, but not the time, correlation properties of the fluctuations.

Hereafter, we use the action-angle variables $J, \Psi$ of the unperturbed ($h(q) = 0$) Hamiltonian (1), which will be assumed to be known, to analyze these evolution equations. The perturbed Hamiltonian $H$ in these variables has the form:

$$H = H_0(J) + V(J,\Psi)\xi(t) \tag{6}$$

where $V(J,\Psi) = h(q(J,\Psi))$ and $H_0(J)$ are known functions.

## 3 Evolution equations.

Both the average density $\bar{f}$ and the correlator $K$ are evolving in time. We will derive the evolution equations for both quantities using basically the conventional techniques of the theory of stochastic differential equations /1/. It had been shown previously /2-4/ that the evolution of the average density obeys the Fokker-Planck equation. However, to the authors' knowledge, the evolution of the density fluctuations has never been studied.

In the action-angle variables, the average density and the correlator are given by: $\bar{f} = \bar{f}(J,\Psi,t)$ and $K = K(J,\Psi,\tilde{J},\tilde{\Psi},t)$. We will also use the notation $\tilde{\Psi} = \Psi + \varphi$ and compressed notations for the phase space coordinates : $(1) = x_{1i} = (J,\Psi)$ and $(2) = x_{2i} = (\tilde{J},\tilde{\Psi})$. Taking the differentially small time increment $\Delta t$, one obtains the derivatives of the average density,

$$\frac{\partial \bar{f}(1)}{\partial t} = \lim_{\Delta t\to 0} \frac{\Delta f(1)}{\Delta t}, \tag{7}$$

and the correlator,

$$\begin{aligned} \frac{\partial K}{\partial t} \;=\; & \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left(\langle \Delta f(1) f(2)\rangle + \langle f(1)\Delta f(2)\rangle + \langle \Delta f(1)\Delta f(2)\rangle\right. \\ & \left. - \bar{f}(1)\langle \Delta f(2)\rangle - \bar{f}(2)\langle \Delta f(1)\rangle - \langle \Delta f(1)\rangle\langle \Delta f(2)\rangle\right), \end{aligned} \tag{8}$$

where the increment of the density $\Delta f = f(t + \Delta t) - f(t)$ can be expressed, due to the conservation of the phase-space density, as

$$\Delta f = \frac{\partial f}{\partial x_i}\Delta x_i + \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_k}\Delta x_i \Delta x_k, \tag{9}$$

where a summation over repeating indices is assumed. The increments of the phase-space variables $\Delta x_i$ in time $\Delta t$ can be obtained from the stochastic equations of motion. The second order terms in $\Delta x$ were kept because of the properties of the white noise, where the average of quadratic terms $\Delta x_i \Delta x_j$ produces terms linear in $\Delta t$.

Substituting equation (9) into equation (8) and making ensemble averages, one finds that the averages of products of $x$'s and $f$'s are factorizable, i.e.

$$\begin{aligned} \langle \Delta f(1) \rangle &= \langle \Delta x_i \rangle \left\langle \frac{\partial f(1)}{\partial x_{1i}} \right\rangle + \frac{1}{2} \langle \Delta x_{1i} \Delta x_{1k} \rangle \left\langle \frac{\partial^2 f(1)}{\partial x_{1i} \partial x_{1k}} \right\rangle, \qquad (10) \\ \langle \Delta f(1) \Delta f(2) \rangle &= \langle \Delta x_{1i} \Delta x_{2k} \rangle \left\langle \frac{\partial f(1)}{\partial x_{1i}} \frac{\partial f(2)}{\partial x_{2k}} \right\rangle, \qquad (11) \end{aligned}$$

etc. This is due to the fact that the increments $\Delta x_i$ depend on the noise $\xi(t)$ only in the time range between $t$ and $t + \Delta t$. One obtains then the evolution equations for the moments of the density in the form:

$$\begin{aligned} \frac{\partial \bar{f}}{\partial t} &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \langle \Delta x_{1i} \rangle \frac{\partial \bar{f}}{\partial x_{1i}} + \frac{1}{2} \langle \Delta x_{1i} \Delta x_{1k} \rangle \frac{\partial^2 \bar{f}}{\partial x_{1i} \partial x_{1k}} \right\} \qquad (12) \\ \frac{\partial K}{\partial t} &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \langle \Delta x_{1i} \rangle \frac{\partial K}{\partial x_{1i}} + \langle \Delta x_{2i} \rangle \frac{\partial K}{\partial x_{2i}} + \frac{1}{2} \langle \Delta x_{1i} \Delta x_{1k} \rangle \frac{\partial^2 K}{\partial x_{1i} \partial x_{1k}} \right. \qquad (13) \\ &\quad \left. + \frac{1}{2} \langle \Delta x_{2i} \Delta x_{2k} \rangle \frac{\partial^2 K}{\partial x_{2i} \partial x_{2k}} + \langle \Delta x_{1i} \Delta x_{2k} \rangle \left( \frac{\partial \bar{f}(1)}{\partial x_{1i}} \frac{\partial \bar{f}(2)}{\partial x_{2k}} + \frac{\partial^2 K}{\partial x_{1i} \partial x_{2k}} \right) \right\} \end{aligned}$$

The moments of $\Delta x$'s that are present in equations (12) and (13) can be computed quite straightforwardly by using the conventional techniques from the theory of stochastic differential equations. The details of this calculation are given in the Appendix. The evolution equation for the average $\bar{f}$ after the substitution of the moments (A4) becomes the conventional Fokker-Planck equation:

$$\begin{aligned} \frac{\partial \bar{f}}{\partial t} &= \left[ \frac{1}{2} \left( \frac{\partial^2 V}{\partial J \partial \Psi} \frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial J^2} \frac{\partial V}{\partial \Psi} \right) + \omega(J) \right] \frac{\partial \bar{f}}{\partial \Psi} - \frac{1}{2} \left( \frac{\partial^2 V}{\partial \Psi^2} \frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial \Psi \partial J} \frac{\partial V}{\partial \Psi} \right) \frac{\partial \bar{f}}{\partial J} \\ &\quad + \frac{1}{2} \left( \frac{\partial V}{\partial J} \right)^2 \frac{\partial^2 \bar{f}}{\partial \Psi^2} + \frac{1}{2} \left( \frac{\partial V}{\partial \Psi} \right)^2 \frac{\partial^2 \bar{f}}{\partial J^2} - \frac{\partial V}{\partial \Psi} \frac{\partial V}{\partial J} \frac{\partial^2 \bar{f}}{\partial \Psi \partial J} \qquad (14) \end{aligned}$$

where $V = V(J, \Psi)$. For the correlator $K$, one obtains an evolution equation that is coupled to the mean $\bar{f}$:

$$\begin{aligned} \frac{\partial K}{\partial t} &= \left[ \frac{1}{2} \left( \frac{\partial^2 V}{\partial J \partial \Psi} \frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial J^2} \frac{\partial V}{\partial \Psi} \right) + \omega(J) \right] \frac{\partial K}{\partial \Psi} - \frac{1}{2} \left( \frac{\partial^2 V}{\partial \Psi^2} \frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial \Psi \partial J} \frac{\partial V}{\partial \Psi} \right) \frac{\partial K}{\partial J} \\ &\quad + \frac{1}{2} \left( \frac{\partial V}{\partial J} \right)^2 \frac{\partial^2 K}{\partial \Psi^2} + \frac{1}{2} \left( \frac{\partial V}{\partial \Psi} \right)^2 \frac{\partial^2 K}{\partial J^2} - \frac{\partial V}{\partial \Psi} \frac{\partial V}{\partial J} \frac{\partial^2 K}{\partial \Psi \partial J} \qquad (15) \end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{1}{2}\left(\frac{\partial^2\tilde{V}}{\partial\tilde{J}\partial\tilde{\Psi}}\frac{\partial\tilde{V}}{\partial\tilde{J}}-\frac{\partial^2\tilde{V}}{\partial\tilde{J}^2}\frac{\partial\tilde{V}}{\partial\tilde{\Psi}}\right)+\omega(\tilde{J})\right]\frac{\partial K}{\partial\tilde{\Psi}}-\frac{1}{2}\left(\frac{\partial^2\tilde{V}}{\partial\tilde{\Psi}^2}\frac{\partial\tilde{V}}{\partial\tilde{J}}-\frac{\partial^2\tilde{V}}{\partial\tilde{\Psi}\partial\tilde{J}}\frac{\partial\tilde{V}}{\partial\tilde{\Psi}}\right)\frac{\partial K}{\partial\tilde{J}}\\
&+\frac{1}{2}\left(\frac{\partial\tilde{V}}{\partial\tilde{J}}\right)^2\frac{\partial^2 K}{\partial\tilde{\Psi}^2}+\frac{1}{2}\left(\frac{\partial\tilde{V}}{\partial\tilde{\Psi}}\right)^2\frac{\partial^2 K}{\partial\tilde{J}^2}-\frac{\partial\tilde{V}}{\partial\tilde{\Psi}}\frac{\partial\tilde{V}}{\partial\tilde{J}}\frac{\partial^2 K}{\partial\tilde{\Psi}\partial\tilde{J}}\\
&+\frac{\partial V}{\partial\Psi}\frac{\partial\tilde{V}}{\partial\tilde{\Psi}}\frac{\partial^2 K}{\partial J\partial\tilde{J}}-\frac{\partial V}{\partial\Psi}\frac{\partial\tilde{V}}{\partial\tilde{J}}\frac{\partial^2 K}{\partial J\partial\tilde{\Psi}}-\frac{\partial V}{\partial J}\frac{\partial\tilde{V}}{\partial\tilde{\Psi}}\frac{\partial^2 K}{\partial\Psi\partial\tilde{J}}+\frac{\partial V}{\partial J}\frac{\partial\tilde{V}}{\partial\tilde{J}}\frac{\partial^2 K}{\partial\Psi\partial\tilde{\Psi}}\\
&+\frac{\partial V}{\partial\Psi}\frac{\partial\tilde{V}}{\partial\tilde{\Psi}}\frac{\partial\bar{f}(1)}{\partial J}\frac{\partial\bar{f}(2)}{\partial\tilde{J}}-\frac{\partial V}{\partial\Psi}\frac{\partial\tilde{V}}{\partial\tilde{J}}\frac{\partial\bar{f}(1)}{\partial J}\frac{\partial\bar{f}(2)}{\partial\tilde{\Psi}}\\
&-\frac{\partial V}{\partial J}\frac{\partial\tilde{V}}{\partial\tilde{\Psi}}\frac{\partial\bar{f}(1)}{\partial\Psi}\frac{\partial\bar{f}(2)}{\partial\tilde{J}}+\frac{\partial V}{\partial J}\frac{\partial\tilde{V}}{\partial\tilde{J}}\frac{\partial\bar{f}(1)}{\partial\Psi}\frac{\partial\bar{f}(2)}{\partial\tilde{\Psi}}
\end{aligned}
$$

where $\tilde{V}=V(\tilde{J},\tilde{\Psi})$.

# 4 Small noise / fast oscillation regime.

## 4.1 Averaged evolution equations.

On a long time scale, and in the small noise / fast oscillations regime, one can average the dependence of all quantities along the unperturbed trajectories $J=\text{const}$, $\Psi=\omega(J)t$. This approximation is well known under the name of "averaging of fast-oscillating variables" in the theory of Fokker-Planck equations (see, e.g. /1/), and was also used in previous studies of the average density diffusion /2-4/. For the Fokker-Planck equation (14), the procedure is technically very simple. One assumes that the density $\bar{f}$ is independent of $\Psi$ by taking both the averages over $\xi$ and $\Psi$ (with the double average notation $\langle\langle ...\rangle\rangle$) for all coefficients. The resulting averaged Fokker-Planck equation becomes the well known diffusion equation /2-4/,

$$\frac{\partial\bar{f}}{\partial t}=\frac{\partial}{\partial J}\left(D_J(J)\frac{\partial\bar{f}}{\partial J}\right), \tag{16}$$

where the diffusion intensity $D_J$ is given by (see Appendix for details),

$$D_J(J)=\frac{1}{2}\langle\langle\frac{(\Delta J)^2}{\Delta t}\rangle\rangle=\frac{1}{2}\sum_n n^2|V_n|^2, \tag{17}$$

where $V_n$ are the harmonic amplitudes in the Fourier expansion of $V$ in the $2\pi$ periodic variable $\Psi$.

The implementation of the same small noise and fast oscillations approximation in the evolution equation (15) for the correlator is somewhat more subtle. We will postulate at this point (and confirm it by the final results) that the correlator $K$ does not depend on the phase $\Psi$ but retains the dependence on the phase difference $\varphi=\Psi-\tilde{\Psi}$. The procedure for the evolution equation derivatition then parallels that for the Fokker-Planck equation: one adds an extra averaging in $\Psi$ in all moments in equation (13) while retaining a dependence on $\varphi$, and assumes $K$ to depend on the phases $\Psi$ and $\tilde{\Psi}$ only through the combination $\varphi = \Psi - \tilde{\Psi}$. Using the moments calculated in the Appendix in the formulas (A6) and (A7), one obtains thus:

$$
\begin{aligned}
\frac{\partial K}{\partial t} &= \left(\omega(J) - \omega(\tilde{J})\right) \frac{\partial K}{\partial \varphi} + \frac{\partial}{\partial J}\left(D_J(J)\frac{\partial K}{\partial J}\right) + \frac{\partial}{\partial \tilde{J}}\left(D_J(\tilde{J})\frac{\partial K}{\partial \tilde{J}}\right) \\
&+ \left(D_\Psi(J) + D_\Psi(\tilde{J})\right)\frac{\partial^2 K}{\partial \varphi^2} + F_J(J,\tilde{J},\varphi)\left(\frac{\partial \bar{f}(J)}{\partial J}\frac{\partial \bar{f}(\tilde{J})}{\partial \tilde{J}} + \frac{\partial^2 K}{\partial J \partial \tilde{J}}\right) \\
&+ F_\Psi(J,\tilde{J},\varphi)\frac{\partial^2 K}{\partial \varphi^2},
\end{aligned}
\tag{18}
$$

where the functions $D_\Psi, F_J, F_\Psi$ appear from the moments in (A6), (A7):

$$
\begin{aligned}
D_\Psi &= \frac{1}{2}\sum_n \left|\frac{\partial V_n}{\partial J}\right|^2 \\
F_J(J,\tilde{J},\varphi) &= \sum_n n^2 V_n(J)\tilde{V}_{-n}(\tilde{J})e^{in\varphi} \\
F_\Psi(J,\tilde{J},\varphi) &= \sum_n \frac{\partial V_n(J)}{\partial J}\frac{\partial \tilde{V}_{-n}(\tilde{J})}{\partial \tilde{J}}e^{in\varphi}
\end{aligned}
\tag{19}
$$

### 4.2 Asymptotic solution for the correlator.

In the absence of noise, the solution of equation (18) for the correlator is trivial as only the first term in the r.h.s. survives. The correlation "decays" or rather decoheres due to the phase-mixing. The general solution is an arbitrary function of $\varphi + (\omega(J) - \omega(\tilde{J}))t$. The time scale of decoherence is $\tau \sim 1/\lambda\sigma$, where $\lambda = \frac{d\omega}{dJ}$ and $\sigma$ is the r.m.s. value of $J$ for the distribution $\bar{f}$. In the presence of small noise, when the diffusion coefficient is $D_J \sim |V|^2$, the characterictic diffusion time is $\tau_d \sim \frac{\sigma^2}{D_J}$. In the limit of small noise, we expect that the decoherence time is much shorter than the diffusion time. Furthermore, the correlation "injection", that is provided by the inhomogeneous term in equation (18), varies only on the slow time scale of diffusion. As a result, a quasistationary equilibrium correlation density will be established that is a balance between the slowly changing "injection" of correlations and their fast decay.

To analyze the quasistationary solution, we drop the time derivative of $K$ in equation (18). Another simplification comes from noticing that the correlator $K$ is the largest at a small spatial scale $q = J - \tilde{J}$, where the "decoherence" term (first term in the r.h.s. of equation (18)) is small. Expanding all the coefficients in the equation (18) to the leading order in $q$ and keeping only the dominant derivatives in $q$ yields:

$$
\lambda q \frac{\partial K}{\partial \varphi} + D_J \frac{\partial^2 K}{\partial q^2} + \sum_n R_n e^{in\varphi} = 0 \tag{20}
$$

where the quantities $\lambda = \frac{d\omega(J)}{dJ}$, $D_J = D_J(J)$ and $R_n = n^2|V_n(J)|^2 \left(\frac{\partial \bar{f}(J)}{\partial J}\right)^2$ depend on $J$ as a parameter. For the nonzero harmonics of $K$ in $\varphi$ one obtains:

$$in\lambda q K_n + D_J \frac{\partial^2 K_n}{\partial q^2} + R_n = 0, \tag{21}$$

where the solution of this equation is given by the Airy functions [6]. For the power spectrum of the fluctuations $\tilde{K_n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dq\, K_n(q) e^{ikq}$ the resulting equation is of the first order:

$$n\lambda \frac{\partial \tilde{K_n}}{\partial k} - D_J k^2 \tilde{K_n} + R_n \delta(k) = 0 \tag{22}$$

and allows an explicit solution:

$$\tilde{K_n}(k) = \begin{cases} -\frac{R_n}{n\lambda} \exp\left(\frac{D_J}{3n\lambda}k^3\right), & \text{if } nk\lambda < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

This is the central result of our analysis. The "correlation radius " of fluctuations $q_c$ (the inverse characteristic wavelength) is seen from this formula to be $q_c \sim \left(\frac{D_J}{|3n\lambda|}\right)^{1/3}$.

A special feature of the spectrum of Eq. (23) is its discontinuity. It is easy to see that this discontinuity is the manifestation of the long $\sim 1/|q|$ "tail" of the correlator $K$. Indeed, for large $|q| \gg q_c$ the second term in Eq. (21) becomes much smaller than the first, and one obtains $1/|q|$ tail. It is possible to obtain a more general expression for the "tail" of $K$ for $|q| \gg q_c$ that is not limited by the condition $|q| \ll J$ by keeping the same terms of the primary evolution Eq. (18) (i.e. the first term in the r.h.s. and the inhomogeneous term) without expanding in $q$. The resulting expression for the "tail" is:

$$K_n(J, \tilde{J}, t) = \frac{iV_n(J)V_n^*(\tilde{J})}{n(\omega(J) - \omega(\tilde{J}))} \frac{\partial \bar{f}(J,t)}{\partial J} \frac{\bar{f}(\tilde{J},t)}{\partial \tilde{J}} \tag{24}$$

The most important quantity characterizing the fluctuations is their intensity, which is the value of the correlator $K$ at $q = 0$, and can be calculated by integrating the spectrum $\tilde{K}_n$. The resulting intensities $P_n = K_n(0)$ are:

$$P_n(J) = \frac{\Gamma(1/3)}{3^{2/3}} \frac{R_n(J)}{(n\lambda(J))^{2/3} D_J^{1/3}(J)}. \tag{25}$$

Thus, the fluctuation intensities are of the order $P_n \sim (D_J/\lambda)^{2/3}$ and will be small for small noise /large nonlinearity.

## 5 Conclusions.

We presented the evolution equation formalism for the same-time correlation function of the density distribution fluctuations in the phase space of a nonlinear oscillator under the influnce of coherent (same for all particles) random driving. For the weak noise/ large nonlinearity of oscillations the fluctuations are small and short-ranged in the energy of oscillations. The fluctuations present themselves as a small "microstructure" on top of a smooth mean distribution function. The mechanism for the loss of correlations is related to the phase-mixing ("decoherence") of density perturbations due to the amplitude-dependent frequency of oscillations. This is not a truly dissipative mechanism, resulting in correlations that indicate a long "memory" of the system as demonstrated by a long $\sim 1/q$ tail for the correlator in the energy difference $q$.

## References


**1.** K.Gardiner, Handbook of stochastic methods, Springer, 1985. G.E.Ulenbeck and L.S. Ornstein, On the theory of the Brownian motion, Phys. Rev. **36**, 823 (1930).

**2.** G.Dome, "Theory of RF Acceleration and RF noise", CERN report No.84-15 *Accelerator School on Antiprotons for Colliding Beams Facilities*, 1984, pp.215-260.

**3.** S.Krinsky, J.M.Wang, Part.Accel. 12 (1982) 107.

**4.** J.Ellison, B.S.Newberger, H.-J.Shih "Effect of RF noise on the SSC beam", Proc. 1991 IEEE Particle Accelerator Conf., p.216.

**5.** S. Hansen, et al., "Longitudinal dilution due to rf noise, IEEE Tran. on Nucl. Sci. **24**, 1452 (1977); D. Boussard, G. Dome, C. Graziani, " The influence of rf noise on the lifetime of bunched proton beams", Proc. 11th Conf. on High Energy Acceleraotrs, CERN, p.620 (1980)

**6.** M. Abramowitz and I. Stegun, "Handbook of Mathmatical Functions", Dover Publications, Inc. New York (1965).


## APPENDIX A

In this Appendix, we calculate the moments of $\Delta x$'s that enter the evolution equations (12) and (13) by employing the conventional methods of stochastic differential equations /1/. We start with using the Hamiltonian equations of motion for the Hamiltonian (6) to present the increments $\Delta J$ and $\Delta\Psi$ in the form:

$$\begin{aligned} \Delta\Psi &= \omega(J)\Delta t + \Delta\Psi_1 + \Delta\Psi_2 \\ \Delta J &= \Delta J_1 + \Delta J_2 \end{aligned} \tag{A1}$$

where $\Delta J_1$, $\Delta\Psi_1$ are the first-order terms:

$$\begin{aligned} \Delta J_1 &= -\frac{\partial V}{\partial \Psi}\int_t^{t+\Delta t} dt'\,\xi(t') \\ \Delta\Psi_1 &= \frac{\partial V}{\partial J}\int_t^{t+\Delta t} dt'\,\xi(t') \end{aligned} \tag{A2}$$

and $\Delta J_2$, $\Delta\Psi_2$ are the second-order ones:

$$\begin{aligned}
\Delta J_2 &= -\left(\frac{\partial^2 V}{\partial\Psi^2}\frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial\Psi\partial J}\frac{\partial V}{\partial\Psi}\right)\int_t^{t+\Delta t} dt' \int_t^{t'} dt''\, \xi(t')\xi(t'') \\
\Delta\Psi_2 &= \left(\frac{\partial^2 V}{\partial J\partial\Psi}\frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial J^2}\frac{\partial V}{\partial\Psi}\right)\int_t^{t+\Delta t} dt' \int_t^{t'} dt'\, dt''\, \xi(t')\xi(t'')
\end{aligned} \tag{A3}$$

Averaging over the $\delta$-correlated random process $\xi$ yields then the following expressions for the same-point moments of $\Delta x$'s (no mixing of 1 and 2 variables) :

$$\begin{aligned}
\langle\Delta J\rangle &= \langle\Delta J_2\rangle = -\frac{\Delta t}{2}\left(\frac{\partial^2 V}{\partial\Psi^2}\frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial\Psi\partial J}\frac{\partial V}{\partial\Psi}\right) \\
\langle\Delta\Psi\rangle &= \omega(J)\Delta t + \langle\Delta\Psi_2\rangle = \omega(J)\Delta t + \frac{\Delta t}{2}\left(\frac{\partial^2 V}{\partial J\partial\Psi}\frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial J^2}\frac{\partial V}{\partial\Psi}\right) \\
\langle(\Delta J)^2\rangle &= \langle\Delta J_1^2\rangle = \Delta t\left(\frac{\partial V}{\partial\Psi}\right)^2 \\
\langle(\Delta\Psi)^2\rangle &= \langle\Delta\Psi_1^2\rangle = \Delta t\left(\frac{\partial V}{\partial J}\right)^2 \\
\langle\Delta J\Delta\Psi\rangle &= \langle\Delta J_1\Delta\Psi_1\rangle = -\Delta t\frac{\partial V}{\partial\Psi}\frac{\partial V}{\partial J}
\end{aligned} \tag{A4}$$

and for the different-point moments:

$$\begin{aligned}
\langle\Delta J\Delta\tilde{J}\rangle &= \langle\Delta J_1\Delta\tilde{J}_1\rangle = \Delta t\frac{\partial V}{\partial\Psi}\frac{\partial\tilde{V}}{\partial\tilde{\Psi}} \\
\langle\Delta J\Delta\tilde{\Psi}\rangle &= \langle\Delta J_1\Delta\tilde{\Psi}_1\rangle = -\Delta t\frac{\partial V}{\partial\Psi}\frac{\partial\tilde{V}}{\partial\tilde{J}} \\
\langle\Delta\Psi\Delta\tilde{J}\rangle &= \langle\Delta\Psi_1\Delta\tilde{J}_1\rangle = -\Delta t\frac{\partial V}{\partial J}\frac{\partial\tilde{V}}{\partial\tilde{\Psi}} \\
\langle\Delta\Psi\Delta\tilde{\Psi}\rangle &= \langle\Delta\Psi_1\Delta\tilde{\Psi}_1\rangle = \Delta t\frac{\partial V}{\partial J}\frac{\partial\tilde{V}}{\partial\tilde{J}}
\end{aligned} \tag{A5}$$

In the latter expressions, we used the notations $V = V(J,\Psi)$ and $\tilde{V} = V(\tilde{J},\tilde{\Psi})$.

In the small noise/ fast oscillations approximation, the moments have to be averaged over the phase $\Psi$ while keeping the dependence on the phase difference $\varphi = \Psi - \tilde{\Psi}$. Using the Fourier series $V(J,\Psi) = \sum_n V_n(J)e^{in\Psi}$, one obtains:

$$\begin{aligned}
\langle\langle\Delta J\rangle\rangle &= \frac{1}{2}\Delta t\sum_n n^2\frac{\partial|V_n|^2}{\partial J} \\
\langle\langle\Delta\Psi\rangle\rangle &= \omega(J)\Delta t \\
\langle\langle(\Delta J)^2\rangle\rangle &= \Delta t\sum_n n^2|V_n|^2
\end{aligned} \tag{A6}$$

$$
\begin{aligned}
\langle\langle(\Delta\Psi)^2\rangle\rangle &= \Delta t \sum_n \left|\frac{\partial V_n}{\partial J}\right|^2 \\
\langle\langle\Delta J \Delta\Psi\rangle\rangle &= 0
\end{aligned}
$$

Note here that the relation $\langle\langle\Delta J\rangle\rangle = \frac{1}{2}\frac{\partial}{\partial J}\langle\langle(\Delta J)^2\rangle\rangle$ was verified experimentally /5/ and was proven in general for all Hamiltonian system with random noise /2,3/. Similarly, the phase averaging for the different-point moments yields the expressions:

$$
\begin{aligned}
\langle\langle\Delta J \Delta\tilde{J}\rangle\rangle &= \Delta t \sum_n n^2 V_n \tilde{V}_{-n} e^{in\varphi} \\
\langle\langle\Delta J \Delta\tilde{\Psi}\rangle\rangle &= 0 \\
\langle\langle\Delta\Psi \Delta\tilde{J}\rangle\rangle &= 0 \\
\langle\langle\Delta\Psi \Delta\tilde{\Psi}\rangle\rangle &= \Delta t \sum_n \frac{\partial V_n}{\partial J}\frac{\partial \tilde{V}_{-n}}{\partial \tilde{J}} e^{in\varphi}.
\end{aligned}
\tag{A7}
$$